Qualifying Exam Complex Analysis August, 2018

1. Let a, b, c be three distinct points in $\mathbb{C}$. Show that a necessary and sufficient condition for them to form (vertices of) an equilateral triangle is that

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

2. True or false: if a function f is continuous on $\{z \in \mathbb{C} : |z| \leq 1\}$ and analytic on $\{z \in \mathbb{C} : |z| < 1\}$, then there is $\varepsilon > 0$ such that f extends to a function analytic on $\{z \in \mathbb{C} : |z| < 1 + \varepsilon\}$? Give a proof or counterexample (with explanation).

3. Let f be a nonconstant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

4. Let $|a| < 1$ be a complex number. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - a|^2}.$$

5. Find a one-to-one conformal map of the semi-disc $\{z \in \mathbb{C} : \operatorname{Im} z > 0, |z| < 1\}$ onto the half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.